11

Representative agent models in cashless economies

Macroeconomic models have traditionally been constructed from aggregate demand and supply relationships. These relationships are in turn justified by 'micro-foundations': that is to say, it is shown that people or firms (agents) maximizing their utility or expected utility would on average behave according to these relationships. To obtain aggregate from individual behaviour, assumptions have been made about the distribution of preferences and technological idiosyncracies: often, appeal can be made to the central limit theorem, which says that the mean of N independent random variables has a normal distribution, with a standard deviation equal to that of the component variables divided by $\sqrt{N}$.

Nevertheless, the parameters of these relationships are only loosely tied to the parameters of the preferences and technology of individual agents. We saw this earlier when discussing Lucas' critique (Lucas, 1976) of econometric models: certain of the model's parameters will shift under changes of policy regime or other exogenous variable processes. Yet we would like to have models which can be used to evaluate the effects of just such changes. This argues the need for models whose parameters are solely those of preferences and technology, so-called 'deep structure'.

In response to this need, a variety of models have been produced which make dramatically simple assumptions about preferences and technology, in order to make a complete treatment practicable. Often technology is reduced to stochastic per capita endowments of a single consumption good and consumers are treated as identical representative agents.

These models are obviously no use for traditional forecasting and policy analysis where we try to predict and control particular sequences of events. But this is not the use for which they are intended. Rather it is to model an economy in the sense of mimicking and so understanding its (average) time-series properties when shocked; from a policy viewpoint, this should guide us towards policies which improve those properties. This is a long term perspective. With it tends to go the view that economists and policy makers should only be concerned with these average properties over the long term and not with short term sequences of events.

There are horses for courses. In practice, whether they ought to be or not, economists are called upon to help in both the short- and the long-term aspects of problems. Provided they are honest about the shortcomings of the tools they use in both contexts, we can see no objection to them earning their living doing both. For this reason, we have presented the more traditional models already and now proceed to give an account of these deep structure models. Inevitably, it is too short to be more than an introduction (see Sargent, 1987, for a fuller treatment).

THE BASIC STRUCTURE OF A REPRESENTATIVE AGENT MODEL

In these models the representative household maximizes expected utility subject to its budget constraint; the government spends, levies taxes and prints money subject to its budget constraint; and markets clear, imposing general equilibrium. From this structure it is possible to derive relationships between the stochastic shocks and macroeconomic outcomes, such as consumption, interest rates and prices.

Consumer Maximization

Take first the household's decision. In a non-stochastic world it would typically be assumed to maximize a time-additive utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

c_t is consumption, and u is a well-behaved utility function with positive and diminishing marginal utility of consumption. Let its budget constraint be:

$$A_{t+1} = R_t(A_t + y_t - c_t) \tag{2}$$

where A_t is wealth at the beginning of period t, R_t is the interest rate (gross, inclusive of capital repayment), and y_t is income. A_0 and y_t are given.

Using the Lagrangean method, we can write the maximand as:

$$L = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + ... + \mu_0[A_1 - R_0(A_0 + y_0 - c_0)] \\ + \mu_1[A_2 - R_1(A_1 + y_1 - c_1)] + \mu_2[A_3 - R_2(A_2 + y_2 - c_1)] + ... \quad (3)$$

yielding the first-order conditions with respect to the consumer's choice variables $c_0,\ c_1, ...;\ A_1,\ A_2, ...$ as :

$$0 = \frac{\delta L}{\delta c_0} = u'(c_0) + \mu_0 R_0;\ 0 = \frac{\delta L}{\delta A_1} = \mu_0 - \mu_1 R_1$$

$$0 = \frac{\delta L}{\delta c_1} = \beta u'(c_1) + \mu_1 R_1;\ 0 = \frac{\delta L}{\delta A_2} = \mu_1 - \mu_2 R_2 \quad (4)$$

where $u' = \frac{\delta u}{\delta c}$.

Equation (4) yields a string of relationships between the marginal utility of consumption in one period and the next:

$$u'(c_t) = \beta R_t u'(c_{t+1}) \quad (5)$$

The household equates its marginal rate of transformation

$$u'(c_t)/\beta u'(c_{t+1})$$

with the gross rate of interest, a result illustrated in figure 11.1.

As figure 11.1 and equation (5) suggest, one can split up the consumer's problem into a sequence of two-period decisions. Given $(A_t + y_t)$, he decides c_{t+1} relative to c_t: that is, he can either decide c_{t+1} if he has already decided on c_t or c_t if he must consume c_{t+1}. This splitting up of the decision problem is known as dynamic programming. In dynamic programming, each period's consumption is first solved given the last period's consumption, wealth and income: finally the initial consumption level is set so that all assets are ultimately consumed.

For many purposes we shall only need to consider the first stage of the decision process. Occasionally, dynamic programming solutions are presented backwards from the future (that is, consumption in one period is solved given consumption in the next period): this method is convenient if there is some fixed terminal point from which one can work back (at which for example the consumer dies, leaving a fixed or no bequest). But it is obviously only a presentational matter whether one period's consumption is seen as depending on last period's or next period's.

The consumer decides what to do one period at a time, and in principle he can recompute his decision for the next period when it comes: he decides c_0 this period (with a plan for c_1, $c_2, \ldots$), next period he decides

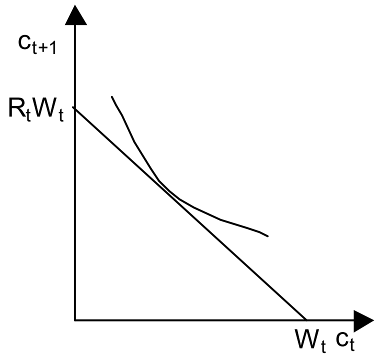

Figure 11.1: Intertemporal consumer choice

c_1 (given c_0, then in the past, and with a plan for $c_2,\ c_3, \ldots$). Of course in this problem with no stochastic shocks he will always stick to his original plan (this is not to be confused with the time inconsistency of policy makers who can influence other people's decisions and then recompute: our consumer only affects himself). He might as well decide at the start on his consumption plan and just carry it out without further thought.

We now turn to a stochastic environment, where in each period a particular shock is realized. The consumer will have decided on consumption in the previous period based on his expectation across all possible shocks. He will also have a plan for his consumption in future periods; this will be a contingency plan, in which his consumption will depend on which shocks occur. Since he does not actually have to decide irrevocably on future consumption until the period involved, this contingency approach is the optimal one: he maintains his flexibility until the last possible moment. Then as the shocks are realized, he picks the relevant branch of his contingency plan. Figure 11.2 illustrates.

We can think of this equivalently as the consumer either recomputing his best expected plan each period or as computing at the start a total contingency plan and carrying it out as the shocks are realized: the point is that the consumer is making use of his potential flexibility in the face of shocks by deferring decisions on actual consumption until he has to take them.

The consumer is assumed to maximize expected utility in this envi-

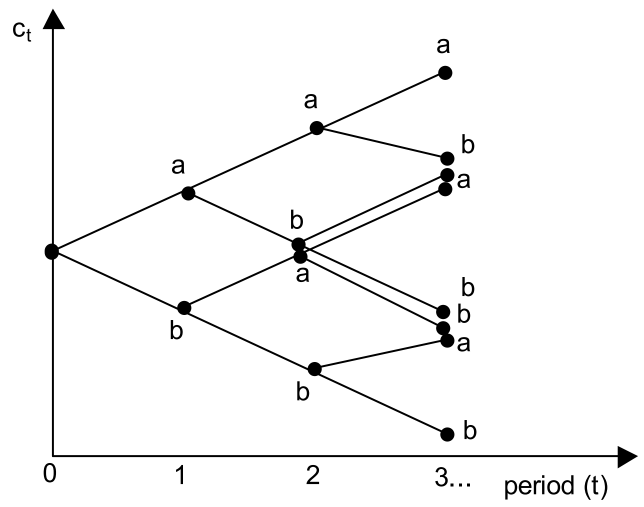

Figure 11.2: Contingency plan for consumption, with the shock each period taking two values (a, b)

ronment, or

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $A_{t+1} = R_t(A_t + y_t - c_t)$, where $R_t = R(1 + \varepsilon_t)$ for example but y_t is a known series. We can now use the principle of dynamic programming and take each period's decision separately.

We can write U_0, substituting c_1 out from the constraints, as:

$$U_0 = u(c_0) + E_0\beta u(R_0[A_0 - c_0] + y_1 - A_2/R_1) + E_0\beta^2 u(c_2) + ... \quad (6)$$

Maximizing this with respect to c_0 (A_0 given) gives:

$$0 = \frac{\delta U_0}{\delta c_0} = u'(c_0) + E_0\beta u'(c_1)(-R_0)$$

or

$$u'(c_0) = E_0\beta R_0 u'(c_1) \quad (7)$$

Analogously at $t = 1$, the consumer maximises U_1 (A_1 given) to obtain:

$$u'(c_1) = E_1\beta R_1 u'(c_2) \quad (8)$$

and in general:

$$u'(c_{t)} = E_t \beta R_t u'(c_{t+1}) \tag{9}$$

which is the expected analogue of (5). By the law of iterated expectations it follows that:

$$E_0 u'(c_t) = \beta E_0 R_t u'(c_{t+1}) \tag{10}$$

We can reach this general result more compactly by using expected Lagrangeans, taking the expectations operator through the Lagrangean multipliers.

Let the consumer be faced with a given realization A_t; then he can choose $c_t,\ c_{t+1},\ A_{t+1}$ to maximise (at t) U_t subject to the constraint, or the expected Lagrangean,

$$\begin{aligned} L = U_t + E_t\{\mu_t(A_{t+1} - R_t[A_t + y_t - c_t]) + \\ \mu_{t+1}(A_{t+2-} R_{t+1}[A_{t+1} + y_{t+1-} c_{t+1}]) + ...\} \end{aligned} \tag{11}$$

$$0 = \frac{\delta L}{\delta c_t} = \beta_t u'(c_t) + E_t \mu_t R_t \tag{12}$$

$$0 = \frac{\delta L}{\delta A_{t+1}} = E_t \mu_t - E_t \mu_{t+1} R_{t+1} \tag{13}$$

$$0 = \frac{\delta L}{\delta c_{t+1}} = E_t \beta_{t+1} u'(c_{t+1}) + E_t \mu_{t+1} R_{t+1} \tag{14}$$

Equation (9) follows by substitution.

GENERAL EQUILIBRIUM

Equation (9) can be turned into a pricing formula for an asset. Suppose the asset yields an uncertain dividend, $d_{t+1,}$ and has a current price, p_t, so that

$$R_t = \frac{p_{t+1} + d_{t+1}}{p_t} \tag{15}$$

Then

$$u'(c_t) = E_t \beta (\frac{p_{t+1} + d_{t+1}}{p_t}) u'(c_{t+1})$$

or

$$p_t = \frac{\beta}{u'(c_t)} E_t u'(c_{t+1)}(p_{t+1} + d_{t+1}) \tag{16}$$

Since all consumers are identical, the only way this asset will be held (by anyone and everyone) is for (15) to hold.

One way to create a simple economy with an asset market is to follow Lucas' tree model (Lucas, 1978). Let the only asset be an identical tree, one initially at t belonging to each consumer, who has no other source of income. Let each tree produce non-storable fruit, an all-purpose consumption good, in the quantity d_t. Let the number of trees belonging to each consumer be S_t; the supply of trees per consumer is $\overline{S} = 1$.

Fruit can be exchanged for trees at the price p_t (units of fruit per tree). But Walras' Law implies that if the fruit market is in excess demand, the tree market is in excess supply. Therefore market clearing across the whole economy (trees and fruit) implies that

$$c_t = d_t \tag{17}$$

Equations (15) and (16) therefore constitute our model of the economy, yielding the compact form:

$$p_t = \beta \frac{1}{u'(d_t)} E_t u'(d_{t+1})(p_{t+1} + d_{t+1)} \tag{18}$$

Substituting successively for $E_t p_{t+1}$, $E_t p_{t+2}$, ... and using the law of iterated expectations ($E_t E_{t+i} = E_t$), yields:

$$\begin{aligned} p_t &= E_t \left[\frac{\beta u'(d_{t+1})}{u'(d_t)} d_{t+1} + \beta^2 \frac{u'(d_{t+1}) u'(d_{t+2})}{u'(d_t) u'(d_{t+1})} d_{t+2} + ... \right] \\ &= E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j)})}{u'(d_t)} d_{t+j} \end{aligned} \tag{19}$$

Depending on the form of the utility function, one can solve for p_t as a function of the current and expected d. One convenient case is where $u(c_t) = lnc_t$ so that $u'(c_{t)} = \frac{1}{c_t}$, in which case

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j d_t = \frac{\beta}{1-\beta} d_t \tag{20}$$

In this case the asset price varies with whatever stochastic process drives the harvest of fruit.

THE GOVERNMENT BUDGET CONSTRAINT

In this general equilibrium framework with no money, let us introduce a government issuing debt, bonds, each of which pays next period one unit of the fruit, regardless; b_{t+1} is the number of such bonds outstanding at t (b_t the number issued in $t-1$). Its budget constraint will be

$$g_t - T_t + b_t = b_{t+1}/R_{1t} \tag{21}$$

where g_t = government spending per capita, T_t = taxes per capita (an equal poll tax), R_t = the one-period-ahead rate of return on debt .

For this debt to be held, it must be as with trees that

$$u'(c_t) = \beta E_t R_t u'(c_{t+1}) \tag{22}$$

Since R_t is certain, this implies

$$R_t^{-1} = \beta \frac{E_t u'(c_{t+1})}{u'(c_t)} \tag{23}$$

Now market clearing of the fruit market implies:

$$c_t + g_t = d_t \tag{24}$$

The consumer's budget constraint is

$$d_t S_t - c_t - T_t + b_t + p_t S_t = \frac{b_{t+1}^d}{R_{1t}} + p_t S_{t+1}^d \tag{25}$$

where S_t is his existing holding of trees and S_{t+1}^d is his desired holding of trees for next period.

Using the two budget constraints, (20) and (24), and fruit market clearing, (23), Walras' Law reappears for debt and trees together:

$$\frac{b_{t+1}}{R_t} + p_t S_t = \frac{b_{t+1}^d}{R_t} + p_t S_{t+1}^d \tag{26}$$

Since debt and trees are perfect substitutes at the R_t given by (22) and p_t given by (18), any excess supply of debt (excess demand for trees) is eliminated by an infinitesimal movement in either p_t or $R_{t.}$

The Ricardian equivalence result immediately follows in this model, that taxes are irrelevant, to consumption from (23), and to interest rates and asset prices:

$$R_t^{-1} = \beta \frac{E_t u'(d_{t+1} - g_{t+1})}{u'(d_t - g_t)} \tag{27}$$

$$p_t = \sum_{j=1}^{\infty} \beta^j \frac{E_t u'(d_{t+j} - g_{t+j}) d_{t+j}}{u'(d_t - g_t)} \tag{28}$$

Only the path of GDP and government spending matters. This is because households are infinitely-lived so that the pattern of (lump-sum) taxes does not affect their life-time consumption possibilities or permanent income.

THE PRICING OF CONTINGENT CLAIMS

We saw earlier (equation 15) that the price of an asset paying a stochastic dividend, d_t, was:

$$p_t = \beta E_t \frac{u'(c_{t+1})(p_{t+1} + d_{t+1})}{u'(c_t)} \tag{29}$$

The price and the dividend is in units of the consumption good, 'fruit' or whatever. Now consider a claim which pays out one unit of the consumption good in $t+1$ when the state of the economy, some vector x_{t+1}, has a value between x_0 and x_1. By extension of (28) its value will be:

$$q_t[x_0 \leq x_{t+1} \leq x_1] = \frac{\beta}{u'(c_t)} \int_{x_0}^{x_1} u'(c_{t+1}[x_{t+1}]) f(x_{t+1}, x_t) dx_{t+1} \tag{30}$$

$f(x_{t+1},\ x_t)$ is the probability density function over x_{t+1} (given that x_t has occurred); integrating the area under this function gives the probability of x_{t+1} lying in the range x_0 to x_1. Equation (29) says that the price of a contingent claim is the marginal utility of one unit next period, relative to this period's marginal utility if x_{t+1} lies in the range, times the probability of its lying in that range.

Equation (29) allows any contingent claim to be priced. One simply specifies the range of contingency, evaluates the marginal utility of consumption in that contingency, multiplies by the pay-off in units of the consumption good, and weights each part of the range by its probability. Equation (29) can also be derived directly from the consumer's maximum problem subject to a budget constraint containing the contingent claim: this is left as an exercise for the interested reader.

Contingent claims which nest other contingent claims within them (for example, a claim on two-period ahead consumption given x_{t+2} and x_t) must be consistent with the claims nested within them; otherwise arbitrage opportunities occur. Hence, for example, a claim on consumption two periods ahead must have the same price today as the current price of a one-period ahead claim on consumption two periods ahead times the current price of a claim on consumption one-period ahead. This is exactly analogous to the arbitraging of interest rates of n-period ahead maturity with the n one-period interest rates for the intervening periods.

GOVERNMENT BONDS AS NET WEALTH: MODELS WITH INTERTEMPORAL AND GEOGRAPHIC CONSTRAINTS

We saw above that, subject to the solvency condition, a government could borrow as much or as little as it liked with no effect on the economy, assuming its taxes were lump sum (distortionary taxes are another matter as we showed in chapter 7, where we discussed the Lucas-Stokey optimal tax-smoothing proposition) — this is Ricardian equivalence. Yet a number of authors have been impressed with the role that a government-issued liability could perform by intermediating between people who may not be willing to lend directly to each other for some reason: they will still be willing to lend to a government which in turn may transfer, or lend, to others. Of course, such government bonds will generally affect economic outcomes and be net wealth.

Two main sets of reasons have been advanced why people would be unwilling to lend to some other people: one is death (the young will not lend to the old because the old have no incentive to pay it back after death), the other is geographic isolation (members of one tribe will not lend to members of another if they cannot see them again to reclaim the debt).

OVERLAPPING GENERATIONS MODELS

Samuelson's overlapping generations model, used (see chapter 7) by Barro (1974) with a bequest motive in order to re-establish Ricardian equivalence, has been used extensively by Neil Wallace and his colleagues under the assumption of inter-generational indifference (no bequests) to establish a role for government liabilities.

Suppose all generations are made up of N identical agents, whose income stream in perishable consumption units is $y - \varepsilon$ and ε in their youth and old age respectively: $0 < \varepsilon < y/2$ so that they obtain more income when young than old. Consumption when young and old is respectively $c_t^h(t)$ and $c_t^h(t+1)$ for the hth agent of generation t. Assume there is no investment opportunity other than a loan market, but that the government has no borrowing, tax or spending programme initially. Then we can easily show that there will be no lending by agents when young, l_t^h, at all: the old will not lend (because they will not be alive to be paid back) or borrow (because they cannot pay back when dead).

Assume the consumer maximizes a logarithmic utility function

$$U_t^h(c_t^h(t), c_t^h(t+1)) = \ln c_t^h(t) + \ln c_t^h(t+1) \tag{31}$$

subject to

$$\begin{aligned} y - \varepsilon - c_t^h(t) &= l_t^h \\ \varepsilon + [1 + r(t)]l_t^h &= c_t^h(t+1) \end{aligned} \tag{32}$$

where $r(t)$ is the net rate of interest, to be determined by the clearing of the loan market (this by Walras' Law also clears the goods market). His Lagrangean is therefore:

$$\begin{aligned} J = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu_t^h(t)[y - \varepsilon - c_t^h(t) - l_t^h] \\ + \mu_t^h(t+1)\{\varepsilon + [(1 + r(t)]l_t^h - c_t^h(t+1)\} \end{aligned} \tag{33}$$

The first-order conditions yield:

$$c_t^h(t) = \frac{c_t^h(t+1)}{1 + r(t)} \tag{34}$$

The consumer's life-time constraint is:

$$c_t^h(t) + \frac{c_t^h(t+1)}{1 + r(t)} = y - \varepsilon + \frac{\varepsilon}{1 + r(t)} \tag{35}$$

so that

$$c_t^h(t) = \frac{y - \varepsilon}{2} + \frac{\varepsilon}{2(1 + r(t))} \tag{36}$$

and

$$l_t^h = \frac{y - \varepsilon}{2} - \frac{\varepsilon/2}{1 + r(t)} \tag{37}$$

Market-clearing requires that $\sum_h l_t^h = 0$ and hence l_t^h=0 since all agents are identical. Consequently

$$1 + r(t) = \frac{\varepsilon}{y - \varepsilon} < 1 \tag{38}$$

Negative interest rates are required to induce people to consume all of their youthful income, since no one is available to make a loan to.

Now let the government borrow (a one-period loan) to spend in the first period only, the sum $G(1) = N[\frac{y}{2} - \varepsilon]$; it pays off its debt from taxes (τ) on the young in period T. Its budget constraint is:

$$G(t) + L^g(t) = \sum \tau^h_{t-1}(t) + \sum \tau^h_t(t) + [1 + r(t-1)]L^g(t-1) \quad (39)$$

where $L^g(t)$ is the loan it takes out in t and it starts with $L^g(0) = 0$. In this case, this implies

$$\begin{aligned} G(1) &= -L^g(1) = N\left[\frac{y}{2} - \varepsilon\right]; \\ L^g(t) &= [1 + r(t-1)]L^g(t-1)\ \{T > t \geq 2\} \end{aligned}$$

and

$$\sum_h \tau^h_T(T) = -[1 + r(T-1)]L^g(T-1), \text{ setting } L^g(T) = 0 \quad (40)$$

The loan market equilibrium in period 1 is now:

$$0 = L^g(1) + \sum_h l^h_1 = -N\left[\frac{y}{2} - \varepsilon\right] + N\left[\frac{y-\varepsilon}{2} - \frac{\varepsilon/2}{1 + r(1)}\right] \quad (41)$$

which is only satisfied if $r(1) = 0$. This government intervention has therefore raised interest rates by siphoning off the young's expenditure into loans. Subsequently, the government debt remains constant: $L^g(2) = L^g(1)$, which implies that $r(2) = 0$; and so on until T, when $L^g(T) = 0$ but the tax bill reduces the income of the young, and $0 = \sum_h l^h_t$ gives $1 + r(T) = \frac{\varepsilon}{y/2}$. Thereafter of course the equilibrium gives $r(t) = \frac{\varepsilon}{y-\varepsilon}$ $(t > T)$ as in (37).

The implications for consumption patterns of this intervention are beneficial to every generation except the Tth and beyond. Generations less than T now consume:

$$c^h_t(t) = \frac{y}{2} = c^h_t(t+1) \quad (42)$$

whereas previously they consumed $y - \varepsilon$ and ε respectively in t and $t+1$. Figure 11.3 illustrates the improvement in their welfare. The 45° line yy shows each generation's consumption possibilities when young and old, provided there is some mechanism (like the government loan sequence above) to ensure that when it transfers resources to the old, the same will be done by the next generation to it. Clearly the optimal point is along the 45° line from the origin.

The Tth generation is worse off, as its consumption pattern becomes $\frac{y}{2}$ in youth (because of the tax bill) and in old age. The following generations are of course unaffected.

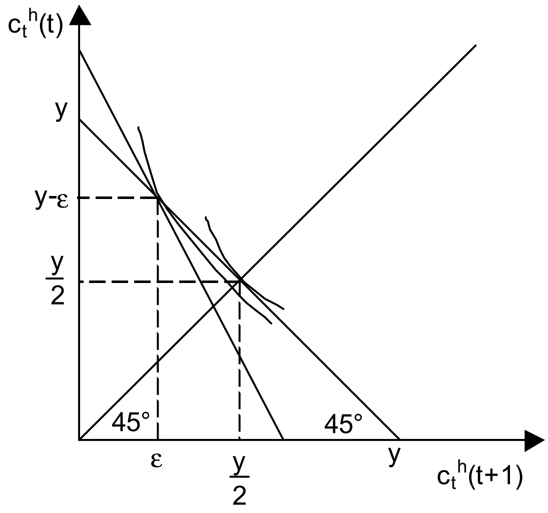

Figure 11.3: The welfare effect of consumption smoothing on the OLG model

This would therefore appear not to be a Pareto-improving policy across all generations. However, in an economy with an infinite life, it can be made so by deferring repayment of the government debt indefinitely: the Tth generation then never arrives. In this case the government bond issue is a source of net wealth to the country, even if the initial output diverted to government spending, $G(1)$, is thrown away!

The gain, to repeat, lies in the ability of government bonds to effect loans from the young generation to the old. The young invest in government debt, the old pay it off; but they do not have to deal with each other directly. Notice that this government loan sequence effects exactly the same as a pay-as-you-go pension scheme, taxing the young by $y - \varepsilon - \frac{y}{2}$ and giving this as a pension to the old; so here a loan market in government debt is all that is needed to enable people to make their own pension provision efficiently.

GEOGRAPHY: THE BEWLEY–TOWNSEND MODEL

A similar point about the usefulness of government debt can be made about two communities, each with uneven income patterns over time, but which are separated in space rather than by time as the above generations are. Let community A people have an income stream in perishable consumption units of $y - \varepsilon$ and ε (a variable harvest say) while community B people have the stream ε and $y - \varepsilon$, in even and odd periods respectively. All A and B people share identical preferences. But the A people will not lend to the B people, or vice versa, because the two groups rarely meet (they may trade but one group having traded, their next trading session is with another group); Bewley (1980) and Townsend (1980) suggest a 'turnpike' with A and B people passing each other in opposite directions, meeting once but never again. The A and B people both live for $T + 1$ periods (where we can allow $T \to \infty$); assume there are N each. It is obvious that A people will consume in even and odd periods $y - \varepsilon$ and ε respectively, B people ε and $y - \varepsilon$; their consumption pattern will be as uneven as their income pattern. As with our overlapping generations model, a government which borrows can smooth their consumption.

Write its budget constraint per capita as:

$$g_t + l_t^g = \tau_1 + (1 + r(t-1))l_{t-1}^g \quad (T \geq t \geq 0, \text{ given } l_{-1}^g = 0) \tag{43}$$

The government levies the same tax, τ_t on everyone; g_t and l_t^g are per capita spending and one-period loans.

Each hth consumer maximizes

$$\sum_{t=0}^{T} \beta' u(c_t^h)$$

subject to

$$c_t^h + l_t^h \leq y_t^h + (1 + r(t-1))l_{t-1}^h \quad (T \geq t \geq 0$$

given $l_{-1}^h = 0$).

The market equilibrium in loans is:

$$\frac{1}{2}l_t^A + \frac{1}{2}l_t^B + l_t^g = 0 \tag{44}$$

Let us for maximum simplicity assume the government merely acts as a lender and borrower, and does not use its tax or spending powers. Then $l_t^g = 0$ (net) for all t and it follows from market equilibrium that

$$c_t^A + c_t^B = y \tag{45}$$

The consumer's optimum in the usual way yields:

$$\frac{u'(c_t^h)}{u'(c_{t+1}^h)} = \beta(1 + r(t)) \tag{46}$$

Hence since from (44) $c_t^B = y - c_t^A$

$$\frac{u'(c_t^A)}{u'(c_{t+1}^A)} = \frac{u'(y - c_t^A)}{u'(y - c_{t+1}^A)} = \beta(1 + r(t)) \tag{47}$$

It follows that $c_t^A = c_{t+1}^A$and $1 + r(t) = \frac{1}{\beta}$: full consumption smoothing with the rate of interest equal to the rate of time preference. Imposing the terminal condition that all loans must be paid off (so that the present values of consumption and income are equal), and letting $T \to \infty$, yields

$$\frac{c^A}{1-\beta} = \frac{y - \varepsilon(1-\beta)}{1-\beta^2} \tag{48}$$

This is the result of equating the present values of A's constant infinite consumption stream and of A's alternating infinite income stream, starting $y - \varepsilon$. Hence

$$c^A = \frac{y - \varepsilon(1-\beta)}{1+\beta}$$

and

$$c^B = \frac{\beta y + \varepsilon(1-\beta)}{1+\beta} \tag{49}$$

This outcome is illustrated in figure 11.4, a box diagram with A's preferences running from the bottom left-hand corner and B's from the top right. Because of positive time preference, their indifference curves have a slope of $-\beta$ along the 45° line between these two corners (compare Figure 11.1 where the slope is -1); A and B agents being identical, their contract curve lies along this 45° line. The autarchic point is at a; the optimum at b is where the budget line going through a with the slope $-(1+r)^{-1} = -\beta$ cuts the contract curve.

The point about the role of public debt here is that it enables *trade* ('geographic smoothing'). With each community made up of infinitely-lived households, then government borrowing (and its tax pattern over time) will make no difference to the pattern of consumption and asset prices — that is, there is Ricardian equivalence. Government debt here creates net wealth only to the extent that it facilitates trade: it is acting like the capital account of the balance of payments. However, if the capital account is already operating, this effect of government debt disappears.

THE REAL BUSINESS CYCLE MODEL

In chapter 3 we referred to the research agenda of 'real business cycles' (RBC), in which the shocks driving the business cycle are identified as

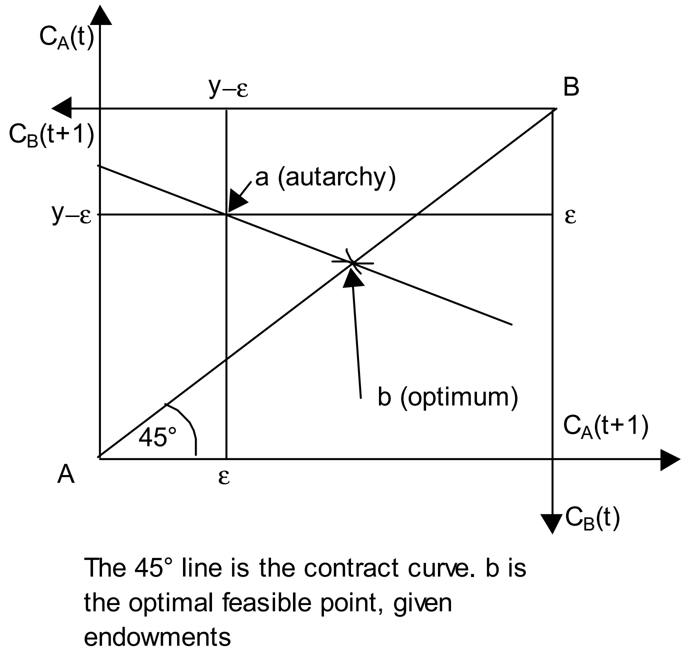

Figure 11.4: The Bewley–Townsend model

coming from productivity and tastes rather than from monetary policy; the reason being that rational agents with access to good up-to-date information will not either sign nominal contracts or suffer information lags about prices sufficient to generate a forward-sloping AS curve — in effect the RBC AS curve is vertical. Even though nowadays we would generally 'augment' such a model with nominal rigidities that would give monetary shocks a role, this uncompromising agenda has produced a large volume of work and insights into the economy's behaviour following Kydland and Prescott (1982), the prime instigators: a large selection is provided by Hartley et al. (1998).

In early post-war macroeconomic thought productivity growth was treated as a smooth and predictable 'deterministic trend', that is a (normally loglinear) time trend; so were changes in tastes that for example through changing labour supply affected output growth. Thus the economy could be thought of as a time trend of 'potential output' with business cycle movements around it:

$$y_t^* = \gamma t + y_0^* \tag{50}$$

The movements around it were seen as due to demand shocks. The RBC model denies this; shocks to potential output are random walks with drift (with possibly additional serial correlation which we ignore for simplicity)

$$\Delta y_t^* = \gamma + \eta_t \tag{51}$$

Here γ is the deterministic trend (the drift), but the key point is that potential output is also constantly being shocked upwards or downwards by permanent changes in tastes (leisure preference) and productivity (for example upwards by innovation, downwards by a new technology that outdates existing capital). If we integrate (50), we obtain:

$$y_t^* = \gamma t + y_0^* + \sum_{i=t-1}^{0} \eta_{t-i} \tag{52}$$

which shows that potential output reflects the cumulant of past shocks to technology and tastes (dominated probably by productivity shocks to the production function).

When one thinks about the matter, it is clear that this rather than (49) must be correct, since technological change is by definition unpredictable and yet once it has occurred its effect is permanent and (unless and until some new change) irreversible. The seminal piece of empirical work that showed (50) fitted the facts better than (49) was Nelson and Plosser (1982).

Given that the progress of potential output is random, this source of shocks can also disturb the economy in the short run, with the assistance of propagation mechanisms. First, investment in the new technology will be spurred by its arrival; the additional capital will take time to build, generating a delayed stimulus to demand. Second, consumer-workers will adjust their consumption and work-plans in response to the new income prospect and real interest movements. The business cycle results from the interplay of these reactions within a market-clearing environment (that is, one where agents are free to make all mutually-beneficial trades).

Plainly there is nothing to stop us adding nominal rigidities to this model; but RBC theorists reject these additions as theoretically ill-founded since people will not wish to subject themselves to additional (money) shocks when they can set prices in relative terms, either by reacting promptly to any changes in money prices or by indexation. RBC modellers claim their models can capture the properties of the business cycle without these additions. A large empirical literature has grown up around these claims, using techniques that are generally different from classical regression: in particular RBC modellers reject forecasting as a valid test of a model. A model should be a mock-up of the economy, to provide insight into how the economy works; thus its 'cyclical properties' should be like those of the economy. This does not imply one can forecast the economy well from a given initial situation, because one cannot forecast the shocks that will occur. RBC modellers favour comparing the unconditional moments of the model's simulations with those of the economy: especially the second moments, the variances and co-variances of key variables like output, employment, interest rates, wages and prices.

We now write down a simple RBC specification. So far our representative agent models have taken the capital stock (e.g. 'trees') as given: income has come as 'endowments' or 'fruit'. Nor have we included labour supply. Essentially we have explored models designed to shed light on particular issues — such as asset pricing and the wealth effects of government borrowing; these models have had both infinitely-lived people, equivalent (as in Barro, 1974) to overlapping generations of mortals who care about their predecessors or their successors or both, and overlapping generations who did not care either way. RBC models draw on both; but the main group assume the former.

So we begin with with a representative household maximizing expected utility from consumption, c_t, and leisure, $1 - L_t$, at time 0:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\{c_t^{\gamma}(a_t[1 - L_t])^{1-\gamma}\}^{1-\rho}}{1 - \rho} \tag{53}$$

where this utility function, usually chosen, is of the Constant Relative Risk Aversion (CRRA) type, with a Cobb-Douglas relation between consumption and leisure (γ is the share of market activity in the total of consumption plus leisure value and a_t is a (leisure) preference shock). All units in the model are per head of population.

The representative firm, owned by the representative household, has a Cobb-Douglas production function with a time-to-build lag between the decision to invest in a capital project and its appearance as capital. The household chooses to consume or to invest in either real bonds or the firm's capital projects. Hence its budget constraint is

$$c_t + I_t + b_t = A_t K_t^{\alpha} L_t^{1-\alpha} - T_t + b_{t-1}(1 + r_{t-1}) \tag{54}$$

where

$$I_t = \sum_{i=1}^{4} k_i s_{t-i+4} \tag{55}$$

$$K_t = (1 - \delta)K_{t-1} + s_{t-4} \tag{56}$$

We have assumed that projects take four periods to complete. Here s_{t-4} for example is the project started at $t-4$, which is then completed in four parts, $k_1 - k_4$, each of these being a proportion. Thus s_{t-4} is completed by t, when it appears in the capital stock, K_t; besides this addition there is depreciation δ. I_t, investment, is the sum of the spending on projects in train: $k_4 s_t,\ k_3 s_{t-1},\ k_2 s_{t-2},\ k_1 s_{t-3}$. It is assumed that a project once started is then taken through to completion: naturally enough because of the large sunk cost of a completed section.

Finally, we have a government which spends on necessary public goods and raises taxes:

$$b_t = G_t - T_t + b_{t-1}(1 + r_{t-1}) \tag{57}$$

Market-clearing in goods is:

$$c_t + I_t + G_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{58}$$

This model is highly nonlinear and cannot be solved analytically, except via a (perhaps unacceptably inaccurate) linear approximation. To illustrate the ideas of such a model we use a simpler set-up in which capital is continuously variable with a one-period lag of installation. This changes the consumer's budget constraint to:

$$c_t + K_t - (1-\delta)K_{t-1} + b_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha} - T_t + b_{t-1}(1 + r_{t-1}) \tag{59}$$

We will also simplify by assuming that households' decision to supply labour is predetermined (perhaps by some decision made a long time before on patterns of education and work); hence L_t itself becomes the preference shock and we drop a_t. The Lagrangean the household maximizes at time 0 now becomes:

$$\Lambda = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\{c_t^{\gamma}(1-L_t)^{1-\gamma}\}^{1-\rho}}{1-\rho} + \lambda_t\{c_t + K_t - (1-\delta)K_{t-1} + b_t - A_t K_{t-1}^{\alpha} L_t^{1-\alpha} + T_t - b_{t-1}(1+r_{t-1})\} \tag{60}$$

The first-order conditions are:

$$0 = E_0\{\beta^t [c_t^{\gamma}(1-L_t)^{1-\gamma}]^{-\rho} \gamma c_t^{\gamma-1} + \lambda_t\} \tag{61}$$

$$0 = E_0\{\lambda_t - \lambda_{t+1}(1+r_t)\} \tag{62}$$

$$0 = E_0\{\lambda_t - (1-\delta)\lambda_{t+1} - \lambda_{t+1} A_{t+1} L_{t+1}^{1-\alpha} \alpha K_t^{-(1-\alpha)}\} \tag{63}$$

These imply the decision for $t = 0$ variables is:

$$c_0 = E_0 c_1 \beta (1+r_0)^{\frac{-1}{1-\gamma}} \left\{ \frac{c_1^{\gamma}(1-L_1)^{1-\gamma}}{c_0^{\gamma}(1-L_0)^{1-\gamma}} \right\}^{\frac{\rho}{1-\gamma}} \tag{64}$$

$$K_0 = \frac{E_0(\alpha\lambda_1 A_1)^{\frac{1}{1-\alpha}} L_1}{\{E_0\lambda_1(r_0+\delta)\}^{\frac{1}{1-\alpha}}} \tag{65}$$

where

$$\lambda_1 = \frac{\gamma\beta(1-L_1)^{-\rho(1-\gamma)}}{c_1^{1-\gamma(1-\rho)}}$$

Taking logs of this we obtain:

$$\ln K_0 = \ln E_0(\alpha\lambda_1 A_1)^{\frac{1}{1-\alpha}} L_1 - \frac{1}{1-\alpha}\ln(r_0+\delta) - \frac{1}{1-\alpha}\ln E_0\lambda_1 \tag{66}$$

Let us now assume that $\Delta \ln A_t = \eta_t$ and $\Delta \ln L_t = \epsilon_t$ where both these errors are normal and iid. We can now make use of the fact that when $\ln Z_t$ is normally distributed with an innovation x_t then approximately

$$\ln E_{t-1} Z_t = E_{t-1} \ln Z_t + 0.5var(x_t) \tag{67}$$

Hence

$$\ln K_0 = \frac{1}{1-\alpha}\ln A_0 + \ln L_0 - \frac{1}{1-\alpha}\ln(r_0+\delta) + \overline{K}_0 \tag{68}$$

where

$$\overline{K}_0 = \frac{1}{1-\alpha} + 0.5\text{var}\{(\alpha\lambda_1 A_1)^{\frac{1}{1-\alpha}} L_1\} - \frac{0.5}{1-\alpha}\text{var}(\lambda_1)$$

is a constant which depends on the variances of the two errors and their covariances with λ.

We can generalise these conditions by letting $0 = t$ (in other words, the period 0 we have been planning from can be any period, t; the plan will then treat 0 as t, 1 as $t+1$ and so on). Hence we can generalise (67) as:

$$\ln K_t = \frac{1}{1-\alpha}\ln A_t + \ln L_t - \frac{1}{1-\alpha}\ln(r_t+\delta) + \overline{K}_0 \tag{69}$$

It follows from the production function that:

$$\ln Y_t = \ln A_t + (1-\alpha)\ln L_t + \frac{\alpha}{1-\alpha}\ln A_{t-1} + \alpha \ln L_{t-1} - \frac{\alpha}{1-\alpha}\ln(r_{t-1}+\delta) + \alpha\overline{K}_0 \tag{70}$$

To solve for interest rates we use the first-order condition for consumption (63) (letting $0 = t$, $1 = t+1$) with (68) for the capital stock and (69) for output, the exogenous process for government spending and substitute them all into the market-clearing equation (57). This is a nonlinear expression in which the expected path of interest rates as seen from period t must ensure that markets clear given public spending plans while consumption is intertemporally smoothed via investment and bond holding. This path (most accurately found via a computer algorithm) will be approximately a saddlepath like our solutions in chapter 2 with forward-looking expectations; plainly there is scope here as there for long drawn-out dynamics from the interaction of adjustment and rational expectations. It is possible to make some rough linear approximations but we do not pursue that here — enough has been done to show that one can obtain solutions for output and the other key macro variables exhibiting both variability and persistence that in principle at least could mimic a real economy — as explored in the empirical literature.

CONCLUSIONS

In this chapter we have examined the behaviour of representative agent models in a cashless society. We have seen that where loan and other contingent claim markets are complete, these claims are priced so that the expected future discounted marginal utility of consumption equals its current marginal utility. Since all agents are identical, consumption must equal (perishable) output in every period. If a government enters the market, then Ricardian equivalence holds: only government spending affects consumption and asset yields. Government debt is irrelevant to real outcomes.

Government debt becomes relevant if some constraint on market completeness prevents optimal loan trades being made between agents. Two such constraints were considered: overlapping generations without a bequest motive where the young will not lend to the old; and communities which are spatially separated so that loans cannot be reliably recovered. In both cases a government which borrows to finance a deficit can achieve intermediation between surplus and deficit agents. In the first case, Ricardian equivalence is eliminated; in the second case Ricardian equivalence still holds for the pattern of government borrowing over time but government willingness to borrow and lend enables trade to occur, in effect proxying the capital account of the balance of payments (chapter 13); of course if capital transactions are already possible, this role for government debt evaporates.

We ended the chapter by setting up a Real Business Cycle model whose aim is to mimic the economy's cyclical properties without appealing to anything other than maximizing behaviour by entirely rational and well-informed agents — it is assumed that they either are well-informed about general price movements or can easily index their own wages and prices to general prices. In this model government policy only matters for the business cycle to the extent that government spending fluctuates: there is Ricardian equivalence so that tax rates are irrelevant and monetary policy is entirely ineffective. The literature exploring this model empirically has not surprisingly generated considerable controversy which continues on many fronts.

APPENDIX 11A THE TECHNIQUE OF DYNAMIC PROGRAMMING

Suppose we write a maximization problem at time $t = 0$ as follows:
Maximize at $t = 0$

$$M = \sum_{0}^{T} \beta^t r(x_t, u_t) + \beta^{T+1} R(x_{T+1}) \tag{1}$$

where (x_0 is given)

$$x_{t+1} = g(x_t, u_t) \tag{2}$$

The r function is the value or return produced in period t; R is the terminal r function, the value in the last period, which the agent is assumed to be unable to affect with his instrument from $T + 1$ onwards. x_t is the state (e.g. of the economy) variable; it may be a vector of variables but we will treat it for simplicity as a single variable. u_t is the variable to be used as the instrument of maximization. The g function is the model of the economy relating the state at $t + 1$ to the previous state and the instrument. There is no uncertainty.

The dynamic programming method is to maximize M in two-period segments, taking the results from other periods as being already maximized. It thus breaks down the problem into T problems. For convenience, start with segment T:
Maximize

$$\beta^T [r(x_T, u_T) + \beta R(x_{T+1})] \tag{3}$$

This takes x_T as given (in effect by separate maximization of previous segments) and, since there is no u_{T+1} by assumption, it must be maximized by choice of u_T. This gives rise to the following first-order condition:

$$0 = \frac{\partial r(x_T, u_T)}{\partial u_T} + \beta \frac{\partial R_{T+1}}{\partial x_{T+1}} \frac{\partial g(x_T, u_T)}{\partial u_T} \tag{4}$$

Since r, g and x_{T+1} are all functions of x_T and u_T, this gives us a maximizing solution for u_T in terms of x_T:

$$\widehat{u_T} = h_T(x_T) \tag{5}$$

Denote the maximizing value of x_{T+1} correspondingly $\widehat{x}_{T+1}$. Then our T segment becomes:

$$[r(x_T, \widehat{u}_T) + \beta R(\widehat{x}_{T+1})] = V_T(x_T) \tag{6}$$

$V_T(x_T)$ is the 'value function' as seen at time T; by this is meant the function which, given x_T, extracts the most value from *subsequent* uses of the instrument (i.e. here u_T).

We must now repeat the operation for the $T-1$ segment:

$$[r(x_{T-1}, u_{T-1}) + \beta V_T(x_T)] \tag{7}$$

where $V_T(x_T)$ is the present value at T of the last, T, segment since this part is what needs to be maximised at $T-1$, having had its maximizing u_T chosen (and so has been reduced, after solving for u_T in terms of x_T, to an expression solely in x_T). Thus we maximise at $T-1$ an expression that already allows for the effect of future maximization at T. Again we obtain the first-order condition:

$$0 = \frac{\partial r(x_{T-1}, u_{T-1})}{\partial u_{T-1}} + \beta \frac{\partial V_T}{\partial x_T} \frac{\partial g(x_{T-1}, u_{T-1})}{\partial u_{T-1}} \tag{8}$$

This again solves for:

$$\widehat{u}_{T-1} = h_{T-1}(x_{T-1}) \tag{9}$$

Again we can rewrite the $T-1$ segment as:

$$[r(x_{T-1}, \widehat{u}_{T-1}) + \beta V_T(\widehat{x}_T)] = V_{T-1}(x_{T-1}) \tag{10}$$

$V_{T-1}(x_{T-1})$ analogously is the value function seen at $T-1$; plainly it includes the value function at T within it, discounted by β. We may continue backwards along the time segments obtaining:

$$\begin{aligned} [r(x_{T-2}, \widehat{u}_{T-2}) + \beta V_{T-1}(\widehat{x}_{T-1})] &= V_{T-2}(x_{T-2}) \\ [r(x_{T-3}, \widehat{u}_{T-3}) + \beta V_{T-2}(\widehat{x}_{T-2})] &= V_{T-3}(x_{T-3}) \qquad (71) \\ &\vdots \end{aligned}$$

$$[r(x_0, \widehat{u}_0) + \beta V_1(\widehat{x}_1)] = V_0(x_0) \tag{11}$$

$V_0(x_0)$ is the maximised value of M, that is, when the whole path of x_1, x_2, , x_{T+1} has been maximized by the path of u_0, u_1, .., u_T.

This segment-by-segment technique is highly convenient for thinking about uncertainty. It is usual to assume that at each period the agent can change his current and future instrument settings. If we think of him maximizing the expected value of (1), then he will at T choose a $\hat{u}_T$ that maximizes the expected value of (4), and (6) will be the discounted value function, as expected at T; this will be his contingency plan for T. $V_0(x_0)$ will then be the expected value function at $t = 0$; and within it will be nested expectations at $t = 1,\ t = 2,\ ...,\ t = T$. We can then use the law of iterated expectations to convert the expression into an expected value at $t = 0$.

Reverting to the case of no uncertainty, we can see that it implies in (9) an optimising rule relating the instrument to the state. Plainly if we can find this rule, then we can substitute it into M in (1) and so find the value function, the maximum value our agent can obtain. Such a rule, which in general will not be time-invariant, would be:

$$\widehat{u}_T = h_T(x_T) \tag{12}$$

In the case of uncertainty, this rule is to be found from the expected first-order condition and it becomes a rule for a contingency plan relating the planned instrument to whatever the state then turns out to be: thus $\hat{u}_T$ becomes the contingent plan value of u_T. If we take expectations of (12), it will give the expected instrument as a function of the expected state.

In practice, with models of any complexity there are no available analytical techniques for finding the contingent-plan rule or the value function: the problem has to be solved numerically via the computer. In this respect, the situation is worse than for the linear rational expectations models we mainly considered in chapters 1-7; for these the analytic techniques exist and can be written down, even if in practice they too are usually found by the computer. For dynamic programming problems the form of the contingent rule has to be guessed in the first place; to minimize complexity, set-ups are often converted into linear models with quadratic maximands, for which the form of the rule is known. However, most of the representative agent models being used in modern research are highly non-linear; so such a conversion carries a cost in brutal approximation.

To gain an understanding of solution paths the computer will generate, it is helpful to work through a simple set-up.

Let a household maximize

$$\sum_{t=0}^{T} \beta^t \ln c_t \tag{13}$$

subject to

$$A_{t+1} = R_t(A_t - c_t) \tag{14}$$

$A_{T+1} = 0; A_0,\ R_t(t = 0,\ 1, \ldots,\ T)$ are given. R_t is the gross real rate of interest.

Here the household has a finite life and will leave no bequest. We keep its problem to the simple one of deciding how to spend an initial stock of wealth, A_0, through its lifetime. In terms of our previous notation, we define the state as A_t and the instrument, u_t, as the amount of wealth unconsumed in period t, or 'savings' for short: $u_t = A_t - c_t$. Thus $A_{t+1} = R_t u_t$.

Starting with the last period, T, plainly there will be no savings: $0 = A_{T+1} = R_T(u_T)$ so that

$$c_T = A_T \tag{15}$$

(a corner solution where all remaining assets are consumed).

The value function at T is therefore:

$$V_T(A_T) = \ln c_T = \ln A_T = \ln R_{T-1} u_{T-1} \tag{16}$$

We now maximize the $T - 1$ segment:

$$\ln(A_{T-1} - u_{T-1}) + \beta \ln R_{T-1} u_{T-1} \tag{17}$$

with respect to u_{T-1}.

The first-order condition is (in the standard way equating the marginal utility of consumption today with β times the marginal utility of consumption the next period times the real interest rate):

$$0 = \frac{-1}{A_{T-1} - u_{T-1}} + \frac{\beta}{u_{T-1}} = \frac{\beta A_{T-1} - (1+\beta) u_{T-1}}{u_{T-1}(A_{T-1} - u_{T-1})} \tag{18}$$

so that

$$u_{T-1} = \frac{\beta}{1+\beta} A_{T-1} \tag{19}$$

that is,

$$c_{T-1} = \frac{1}{1+\beta} A_{T-1} \tag{20}$$

The $T - 1$ value function is now:

$$\begin{aligned} V_{T-1}(A_{T-1)} &= \ln\left(\frac{1}{1+\beta} A_{T-1}\right) + \beta \ln\left(R_{T-1} \frac{\beta}{1+\beta} A_{T-1}\right) \\ &= \beta \ln \beta - (1+\beta)\ln(1+\beta) + \beta \ln R_{T-1} \\ &\qquad + (1+\beta)\ln A_{T-1} \end{aligned} \tag{21}$$

Now maximize the $T - 2$ segment with respect to u_{T-2} :

$$\ln(A_{T-2} - u_{T-2}) + \beta[\beta \ln \beta - (1+\beta)\ln(1+\beta) + \beta \ln R_{T-1}$$

$$+ (1+\beta)\ln(R_{T-2}u_{T-2})]$$

The first-order condition yields analogously:

$$u_{T-2} = \frac{\beta(1+\beta)}{1+\beta(1+\beta)}A_{T-2} \text{ or } c_{T-2} = \frac{1}{1+\beta(1+\beta)}A_{T-2} \tag{22}$$

Proceeding now to maximize the $T-3$ segment with respect to u_{T-3} yields:

$$u_{T-3} = \frac{\beta[1+\beta(1+\beta)]}{1+\beta[1+\beta(1+\beta)]}A_{T-3} \text{ or } c_{T-3} = \frac{1}{1+\beta[1+\beta(1+\beta)]}A_{T-3} \tag{23}$$

Hence we discover that the ratio of savings to wealth starts very close to unity, declines slowly at first, and then plunges sharply in the final years of life. Notice that even in this simple set-up the reaction rule is not constant, the reason being that the end-of-life constraint forces wealth to be completely used up; in order to smooth consumption the share of wealth consumed must rise over the lifetime (figure 11.5).

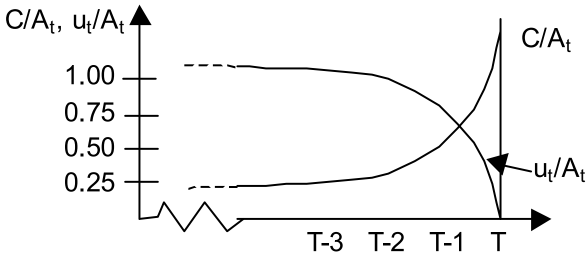

Figure 11.5: Consumption of wealth over lifetime

To find a constant rule requires a problem that does not change over time; an infinite lifetime in this case. If we now change the above problem to an infinite horizon one of maximising at $t = 0$:

$$\sum_{t=0}^{\infty} \beta^t \ln c_t \tag{24}$$

subject to

$$\lim_{t\longrightarrow\infty} A_T = 0; A_{t+1} = R_0(A_t - c_t) = R_0 u_t$$

This assumes that the implied one-period interest rate at $t = 0$, R_0, also applied at all future dates (that is, there is a flat term structure).

We can now guess at a constant reaction function and the corresponding value function. Let us try for V_T the same logarithmic form as the consumption function (since consumption will be related to wealth and therefore welfare at T related to the log of wealth):

$$V_T(A_T) = k \ln A_T + k_0 \tag{25}$$

and

$$u_T = aA_T \tag{26}$$

where k, k_0 and a are unknown and to be found out through the method of undetermined coefficients. To check our guess we derive the first-order condition and then implied value function; if correct, we can solve for the unknown coefficients when we compare these functions with our guessed ones. Using our guess, we maximise with respect to u_T the T-period segment:

$$V_T(A_T) = \ln c_T + \beta V_{T+1}(A_{T+1}) \tag{27}$$

where $V_{T+1}(A_{T+1})$ must be given by (25).

We find the first-order condition of (27) as:

$$u_T = \frac{\beta k}{1+\beta k} A_T \tag{28}$$

We now substitute this into (27) and compare the result with (25):

$$\begin{aligned} V_T(A_T) &= \ln\left(\frac{1}{1+\beta k}A_T\right) + \beta k \ln(R_0 u_T) + \beta k_0 \\ &= \ln\left(\frac{1}{1+\beta k}A_T\right) + \beta k \ln\left(R_0 \frac{\beta k}{1+\beta k}A_T\right) + \beta k_0 \\ &= \beta k \ln \beta k - (1+\beta k)\ln(1+\beta k) + \beta k \ln R_0 + (1+\beta k)\ln(A_T) + \beta k_0 \end{aligned} \tag{29}$$

When set equal to the guessed solution, (25), this yields our undetermined coefficients as:

$$k = \frac{1}{1-\beta}$$

and

$$k_0 = \frac{\beta k \ln \beta k - (1+\beta k)\ln(1+\beta k) + \beta k \ln R_0}{1-\beta} = \frac{\beta \ln \beta + (1-\beta)\ln(1-\beta) + \beta \ln R_0}{(1-\beta)^2}$$

and

$$a = \frac{\beta k}{1 + \beta k} = \beta$$

Hence we have successfully found a constant rule and value function:

$$V_T(A_T) = \frac{1}{1-\beta} \ln A_T + \frac{\beta \ln \beta + (1-\beta)\ln(1-\beta) + \beta \ln R_0}{(1-\beta)^2} \tag{30}$$

$$u_T = \beta A_T \tag{31}$$

$$c_T = (1-\beta) A_T \tag{32}$$

APPENDIX 11B USING DYNAMIC PROGRAMMING TO CHOOSE THE OPTIMAL MONETARY POLICY UNDER OUTPUT PERSISTENCE

In chapter 5 we considered the issue of choosing optimal monetary policy and of time inconsistency. Because the Phillips Curve had no output persistence, the problem could be solved using static optimization. However, suppose that the Phillips Curve exhibits persistence as follows:

$$y_t = \rho y_{t-1} + \alpha(\pi_t - \pi_t^e) + \epsilon_t \tag{1}$$

where π_t^e is the rational expectation of inflation for t formed with $t-1$ information; it is also under commitment the inflation rate chosen and announced by the monetary authority.

Commitment

The set-up is that the central bank has scope to react to shocks- implicitly because the wage contract underlying this Phillips Curve is longer than the publication/reaction time to the shock. Using the usual loss function the value function under commitment is:

$$V(y_{t-1)} = Max(wrt\pi_t, \pi_t^e)E_{t-1}\{-0.5(\pi_t - \pi^*)^2 - 0.5\lambda(y_t - y^*)^2 + \beta V(y_t)\} \tag{2}$$

The value function form that works for the quadratic loss is also quadratic:

$$V(y) = \gamma_0 + \gamma_1 + 0.5\gamma_2 y^2 \tag{3}$$

Under commitment there is an additional constraint that the policy instrument choice at $t-1$ is also to be rationally expected because it will be followed through:

$$E_{t-1}\pi_t = \pi_t^e \tag{4a}$$

and also

$$\pi_t = E_{t-1}\pi_t + b\epsilon_t \tag{4b}$$

which uses the property of rational expectations and b is an undetermined coefficient (here it will be the chosen maximising response of monetary policy to the supply shock).

Hence maximise

$$L_t(= \{-0.5(\pi_t - \pi^*)^2 - 0.5\lambda(y_t - y^*)^2 + \beta V(y_t)\} + \mu(\pi_t - b\epsilon_t - \pi_t^e)$$

where we have used the combined constraint of (4a) and (4b)), to obtain the two first order conditions, the usual one in π_t and then the second $0 = \frac{\partial E_{t-1} L_t}{\partial \pi_t^e}$ respectively:

$$\pi_t = \pi^* - \alpha(\lambda - \beta\gamma_2)[\rho y_{t-1} + \alpha(\pi_t - \pi_t^e) + \epsilon_t] + \alpha\lambda y^* + \beta\gamma_1\alpha + \mu \quad (5)$$

and

$$\mu = \alpha(\lambda - \beta\gamma_2)\rho y_{t-1} - \alpha\lambda y^* - \beta\gamma_1\alpha \quad (6)$$

Substituting for the Lagrange multiplier, μ, from (6) into (5) and taking expectations at $t-1$ yields (via the commitment constraint):

$$E_{t-1}\pi_t = \pi^* = \pi_t^e \quad (7)$$

Hence (5) now yields the optimal inflation rule under commitment:

$$\pi_t = \pi^* - \frac{\alpha^2(\lambda - \beta\gamma_2)}{1 + \alpha^2(\lambda - \beta\gamma_2)}\epsilon_t \quad (8)$$

To find the value function, one could equate (3) for lagged output with the RHS of (2) once the optimal values of π_t and y_t are substituted in the RHS. But as we are only interested in γ_2 in this case we can take the first derivative of each side of (2) with respect to y_{t-1} and equate these expressions (the first derivative of two sides of an equality are equal via the 'envelope theorem'). Differentiating the LHS of (2) yields $\gamma_1 + \gamma_2 y_{t-1}$. The RHS of (2), remembering that once optimised we take expectations of it, is differentiated as:

$$\rho^2(\lambda - \beta\gamma_2)y_{t-1} + \rho\lambda y^* + \rho\beta\gamma_1$$

from which it follows, equating the LHS derivative with the RHS derivative, that:

$$\gamma_1 = \frac{\rho\lambda y^*}{1 + \rho\beta}$$

and

$$\gamma_2 = \frac{-\rho^2\lambda}{1 - \beta\rho^2}$$

Hence:

$$\pi_t = \pi^* - \frac{\alpha\lambda}{1 + \alpha^2\lambda - \beta\rho^2}\epsilon_t$$

Notice that the supply shock response in now larger because of the need to stabilise future output which will now be affected by the persistence term.

Discretion

Now the problem is

$$Max(wrt\ \pi_t)\{-0.5(\pi_t - \pi^*)^2 - 0.5\lambda(y_t - y^*)^2 + \beta V(y_t)\} \quad (9)$$

where now π_t^e is simply the rational expectation of t-inflation formed at $t-1$.

Using the same value function, the first-order condition yields:

$$\pi_t = \frac{1}{1+\alpha^2(\lambda - \beta\gamma_2)}\{\pi^* + \alpha^2(\lambda - \beta\gamma_2)\pi_t^e - \alpha(\lambda - \beta\gamma_2) [\rho y_{t-1} + \epsilon_t] + \alpha\lambda y^* + \beta\alpha\gamma_1\} \quad (10)$$

Taking expectations yields:

$$E_{t-1}\pi_t = \pi^* - \alpha(\lambda - \beta\gamma_2)[\rho y_{t-1}] + \alpha\lambda y^* + \beta\alpha\gamma_1 \quad (11)$$

Hence

$$\pi_t = \pi^* - \alpha(\lambda - \beta\gamma_2)[\rho y_{t-1}] + \alpha\lambda y^* + \beta\alpha\gamma_1 - \frac{\alpha(\lambda - \beta\gamma_2)}{1+\alpha^2(\lambda - \beta\gamma_2)}\epsilon_t = a - b\epsilon_t - cy_{t-1} \quad (12)$$

where a, b and c are implicitly defined.

Now set $\frac{\partial V(y_{t-1})}{\partial y_{t-1}} = \gamma_1 + \gamma_2 y_{t-1}$ (the derivative of the supposed value function wrt y_{t-1})

$$V(y_{t-1}) = E_{t-1}\{-0.5(a - \pi^* - b\epsilon_t - cy_{t-1})^2 - 0.5\lambda(y_t - y^*)^2 + \beta[\gamma_0 + \gamma_1 y_t + 0.5\gamma_2 y_t^2]\} \quad (13)$$

where $y_t = \rho y_{t-1} + [1 - \alpha b]\epsilon_t$

Hence

$$\frac{\partial V(y_{t-1})}{\partial y_{t-1}} = c(a - \pi^*) + \rho\lambda y^* + \rho\beta\gamma_1 - \{c^2 + \rho^2[\lambda - \beta\gamma_2]\}y_{t-1} \quad (14)$$

Equating coefficients between this and the supposed value function derivative yields:

$$\gamma_1 = c(a - \pi^*) + \rho\lambda y^* + \rho\beta\gamma_1$$

and

$$\gamma_2 = -\{c^2 + \rho^2[\lambda - \beta\gamma_2]\}$$

For values where the model is well-behaved the lowest negative root of this quadratic is relevant (see Svensson, 1997) and we obtain the optimizing value of $\widehat{c}$ (and $\widehat{\gamma_2}$). This can then be substituted into the expression for γ_1 to yield:

$$\gamma_1 = \frac{\lambda y^*(\rho + \alpha\widehat{c})}{1 - \beta(\rho + \alpha\widehat{c})}$$

The basic point is that there is an inflation bias $= \frac{\lambda y^*}{1-\beta(\rho+\alpha\widehat{c})} + \widehat{c}y_{t-1}$. Also the response to the current supply shock is excessive because the future inflation bias also depends on today's output; so it needs to be stabilised more strongly. To eliminate these twin problems the monetary authority must be prevented from following a (lagged) feedback rule, since just as in the Sargent and Wallace ineffectiveness result such a feedback component is fully anticipated by wage/price setters and impounded into $E_{t-1}\pi_t$. However, provided the feedback rule is off 'current' output, there is no bias and the optimal stabilization can be achieved; implictly the justification for a feedback off current output is that there are long-term contracts so that the authorities can react to events before the wage/price setters. Hence this set-up can be considered as a simplified overlapping-contract Phillips Curve, where in effect the wage/price setters all contract simultaneously at the start of the period.

Svensson (1997) considers ways this can also be achieved through Walsh contracts, Rogoff 'twisting of preferences', altering the inflation target and the output target. It is obvious that the Walsh contract must be state-contingent to eliminate the lagged output element in the inflation bias. The same is true of the output target; it must now be equal to the 'short-run natural rate', ρy_{t-1}.

However, altering the inflation target to be state-contingent, while it removes the inflation bias, does not restore the optimal response to the supply shock. Because it introduces a remainder term (the square of output appears now additionally because it enters the inflation target) into the utility function, it leads to an over-strong response to the shock, over-stabilising output. In this case the 'intriguing' result occurs that if the authority is made more 'weight-conservative' together with a state-contingent inflation target, the conservativeness can offset this over-reaction, so that the optimal result is restored.

The basic point remains that in practice the central bank must be induced to react only to current shocks and not to lagged information already incorporated into people's contract decisions. This does in practice then require a decision on the relevant length of the 'current' period; new information arriving within this period should be reacted to, previous not. On the logic of long-term contracts overturning the Sargent-Wallace result (Minford and Peel, 2001), this should be the longest period for which nominal contracts are written.